Conservation laws for mechanical systems with unilateral holonomic constraints*

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Abstract A general approach to the construction of conservation laws for mechanical systems with unilateral holonomic constraints by finding corresponding integrating factors is presented. The definition of integrating factors for the Routh equations of motion of the systems is given, and the necessary conditions for the existence of conserved quantities of the unilateral holonomic constraint systems are studied in detail. The conservation theorem and its inverse for the systems are established, and an example is given to illustrate the application of the results.

Keywords: unilateral holonomic constraint, integrating factor, conservation theorem, Killing equation.

It is very important to find out the conservation laws in a field like mathematics, mechanics, physics or any others. Recently, Mei[1] summarized three methods, i.e. the method of Newtonian mechanics, the method of Lagrange mechanics and the symmetry methods, in arriving at the conservation laws and their developments. According to the properties of forces, Newtonian mechanics established the conservation law of momentum, the conservation law of moment of momentum and the conservation law of mechanical energy[1]; Lagrange mechanics found out the conservation laws directly from the expression of dynamical functions and kinematical relations[1,2]; and the theory of symmetry found out the conservation laws from the intrinsic relations of symmetry and conservation laws[1,3~14].

In 1984, Djukic[15] presented an approach to the construction of conservation laws for nonconservative dynamical systems by finding the corresponding integrating factors of the equations of motion. The approach is an attempt to construct a conservation law in a way similar to the one used in obtaining the energy integral for conservative systems, namely that of multiplying the equations of motion by appropriate integrating factors. This is a development of the method of Lagrange mechanics. Qiao et al. applied the approach to the Raitzin's canonical equations of motion of nonconservative systems[16], the canonical equations of motion of nonholonomic relativistic systems[17], the generalized Hamiltonian canonical equations of motion of dynamical systems in generalized classical mechanics[18], the generalized Hamiltonian canonical equations of motion of variable mass nonholonomic dynamical systems[19] and so on. But all these studies are limited to the canonical equations of motion of dynamical systems with bilateral constraints.

This paper further studies the integrating factors and the construction of conservation laws for mechanical systems with unilateral holonomic constraints. The integrating factors for the Routh equations of motion of the systems are defined. According to the definition, the conserved quantities of the systems with unilateral holonomic constraints are constructed, and the conservation theorem and its inverse for the systems are established.

1 Differential equations of motion of the systems and their integrating factors

Suppose that the configuration of a system is determined by n generalized coordinates q_s ($s=1, \cdots, n$), and the motion of the system is subject to g ideal unilateral holonomic constraints

$$f_\beta(t, \boldsymbol{q}) \geqslant 0 \quad (\beta = 1, \cdots, g). \tag{1}$$

If the sign of inequality in constraints (1) holds strictly, then the system is free from the constraints; and if constraints (1) take the sign of equality, the

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system then is in the constraints. If the system is in the constraints, the virtual displacements must be compatible with these constraints, and we have the supplementary conditions

$$\frac{\partial f_\beta}{\partial q_s}\delta q_s = 0 \quad (\beta = 1, \cdots, g). \tag{2}$$

Then the differential equations of motion of the system can be expressed in the form[11]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \quad (s = 1, \cdots, n), \tag{3}$$

where L is the Lagrangian function of the system, Q''_s the generalized nonpotential forces, λ_β the constraint multipliers, and we have

$$\lambda_\beta \geqslant 0, \quad f_\beta \geqslant 0, \quad \lambda_\beta f_\beta = 0 \quad (\beta = 1, \cdots, g). \tag{4}$$

Note that Eq. (3) is not closed, since the discontinuous velocity changes are produced by the existence of the unilateral constraints (1). The connecting conditions along the constraint hypersurface must be taken into consideration if to make Eq. (3) closed, for example, whether the constraint hypersurface is absolutely smooth and the collision is completely elastic, and etc.

If the system is in the constraints, that is, constraints (1) take the sign of equality, we can express λ_β as the functions of t, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and Eq. (3) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s + \Lambda_s,$$

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_s}. \tag{5}$$

If the system is free from the constraints, that is, the sign of inequality in constraints (1) holds strictly, then Eq. (3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s. \tag{6}$$

Now we present the idea of integrating factors for the equations of motion.

Definition 1. If there exist a set of functions $\xi_s = \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, the following identity is satisfied:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s - \Lambda_s\right)\xi_s$$
$$\equiv \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_s}\xi_s - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\tau - G\right]$$
$$+ \mu_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s - \Lambda_s\right),$$
$$f_\beta = 0, \tag{7}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s\right)\xi_s$$
$$\equiv \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_s}\xi_s - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\tau - G\right]$$
$$+ \mu_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s\right), \quad f_\beta > 0, \tag{8}$$

where τ, G and μ_s are functions of t, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, then, functions $\xi_s = \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are called integrating factors for the differential equations of motion (3) of systems with unilateral holonomic constraints.

2 Theorems of conservation of the systems

Combining Eq. (3) with Eqs. (7) and (8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_s}\xi_s - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\tau - G\right]$$
$$= -\mu_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s - \Lambda_s\right), \quad f_\beta = 0, \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_s}\xi_s - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\tau - G\right]$$
$$= -\mu_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s\right), \quad f_\beta > 0, \tag{10}$$

and the following theorem is obvious:

Theorem 1. If the functions ξ_s are integrating factors for Eq. (3), then the following quantity

$$I = \frac{\partial L}{\partial \dot{q}_s}\xi_s - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\tau - G \tag{11}$$

is a conserved quantity (first integral) for systems (1) and (3) with unilateral constraints.

For a given system (1) (3) with unilateral holonomic constraints, if the functions ξ_s are integrating factors for Eq. (3), then the necessary conditions (9) and (10) must be satisfied for each set of functions ξ_s, τ, G and μ_s. Using Eq. (3), conditions (9) and (10) can be written as

$$\frac{\partial L}{\partial q_s}\xi_s + (Q''_s + \Lambda_s)(\xi_s - \dot{q}_s\tau)$$
$$+ \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\partial L}{\partial t}\tau - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\dot{\tau} - \dot{G}$$
$$+ \mu_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q''_s - \Lambda_s\right) = 0,$$

$$f_\beta = 0, \tag{12}$$

$$\frac{\partial L}{\partial q_s}\xi_s + Q_s''(\xi_s - \dot{q}_s\tau) + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \frac{\partial L}{\partial t}\tau - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\dot{\tau} - \dot{G} + \mu_s\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s''\right) = 0, \quad f_\beta > 0. \tag{13}$$

Obviously, if the set of functions ξ_s, τ, G and μ_s satisfy the necessary conditions (12) and (13), then the set of functions reduces the right hand side of (11) into a constant along the trajectory of the given system (1) (3) with unilateral holonomic constraints. Therefore, we obtain the following theorem.

Theorem 2. For every nonsingular set of functions ξ_s, τ, G and μ_s which satisfies the necessary conditions (12) and (13), there exists a conserved quantity (11) of the given system (1) (3) with unilateral holonoimc constraints.

By integrating Eqs. (12) and (13) or by an ad hoc approach, the set of functions ξ_s, τ, G and μ_s can be obtained. For any particular solution or functional solution[15] of Eqs. (12) and (13), which does not contain any integration constant, a conserved quantity (first integral) of the system with unilateral holonomic constraints can be obtained by Theorem 2.

The key to seek for the conserved quantity of the system by Theorems 1 and 2 lies in how to find out the set of functions $\xi_s = \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\tau = \tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Expanding Eqs. (12) and (13) and splitting it into first-order partial differential equations for ξ_s, t and G, which can be called the generalized Killing equations, we can find out these functions by solving the generalized Killing equations. Since the functions ξ_s, τ and G are independent of $\ddot{q}_s$, we can let the terms including $\ddot{q}_s$ and the other terms excluding $\ddot{q}_s$ equal to zero separately, hence, Eqs. (12) and (13) can be split into a system of $(n+1)$ linear partial differential equations, which take the form as follows:

$$\frac{\partial L}{\partial q_s}\xi_s + (Q_s'' + \Lambda_s)(\xi_s - \dot{q}_s\tau) + \frac{\partial L}{\partial \dot{q}_s}\left(\frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}\dot{q}_k\right) - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q_k}\dot{q}_k\right) + \frac{\partial L}{\partial t}\tau - \frac{\partial G}{\partial t} - \frac{\partial G}{\partial q_s}\dot{q}_s + \mu_s\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial t} + \frac{\partial^2 L}{\partial \dot{q}_s\partial q_k}\dot{q}_k - \frac{\partial L}{\partial q_s} - Q_s'' - \Lambda_s\right) = 0,$$

$$f_\beta = 0; \tag{14}$$

$$\frac{\partial L}{\partial q_s}\xi_s + Q_s''\left(\xi_s - \dot{q}_s\tau\right) + \frac{\partial L}{\partial \dot{q}_s}\left(\frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}\dot{q}_k\right) - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q_k}\dot{q}_k\right) + \frac{\partial L}{\partial t}\tau - \frac{\partial G}{\partial t} - \frac{\partial G}{\partial q_s}\dot{q}_s + \mu_s\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial t} + \frac{\partial^2 L}{\partial \dot{q}_s\partial q_k}\dot{q}_k - \frac{\partial L}{\partial q_s} - Q_s''\right) = 0,$$

$$f_\beta > 0; \tag{15}$$

$$\frac{\partial L}{\partial \dot{q}_s}\frac{\partial \xi_s}{\partial \dot{q}_k} - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\frac{\partial \tau}{\partial \dot{q}_k} - \frac{\partial G}{\partial \dot{q}_k} + \mu_s\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k} = 0,$$

$$k = 1, \cdots, n. \tag{16}$$

Eqs. (14) and (16) or Eqs. (15) and (16) are $(n+1)$ equations with $(2n+2)$ unknown functions ξ_s, τ, G and μ_s, which can be called generalized Killing equations. Because the number of the equations is smaller than the number of unknown functions, the solutions of the equations are not unique, and we can obtain different conserved quantities by the appropriate selection of the functions ξ_s, τ, G and μ_s.

When $\mu_s = 0$ $(s = 1, \cdots, n)$, the generalized Killing equations above are of the same form as the generalized Killing equations[12], which are obtained by the Noetherian theory, and there, function G is a gauge variant function and a one-parameter infinitesimal transformation of time and generalized coordinates is given by

$$t^* = t + \varepsilon\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{17}$$

where ε is an infinitesimal parameter. The form of the conserved quantity in Noetherian theory is also the same as that of the conserved quantity (7). Therefore, the integrating factors ξ_s and the functions τ, G, which are of the fundamental importance in the present approach, possess a very clear meaning in the Noetherian theory.

3 Inverse theorem

Assume that the given unilateral constraint system (1) (3) has a first integral

$$I = I(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \text{const}, \tag{18}$$

therefore, the integral (18) and the corresponding integrating factors ξ_s and the functions τ, G must be compatible with the conditions (12) and (13). Calculating $\frac{\partial G}{\partial \dot{q}_s}$ from Eq. (11), and substituting this result into Eq. (16), we obtain

$$\xi_s = \bar{h}_{sk}\frac{\partial I}{\partial \dot{q}_k} + \dot{q}_s\tau + \mu_s \quad (s = 1, \cdots, n), \tag{19}$$

where $\bar{h}_{sk}h_{kl} = \delta_{sl}$. Let the integral (18) equal to the conserved quantity (11), we have

$$\frac{\partial L}{\partial \dot{q}_s}\xi_s - \left(\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - L\right)\tau - G = I. \tag{20}$$

From (20), we obtain

$$G = \frac{\partial L}{\partial \dot{q}_s}\left(\bar{h}_{sk}\frac{\partial I}{\partial \dot{q}_k} + \mu_s\right) + L\tau - I. \tag{21}$$

Hence, we can obtain the following theorem:

Theorem 3. If the unilateral constraint system (1) (3) has a first integral (18), then the integrating factors ξ_s and functions τ, G, μ_s corresponding to the integral are determined by Eqs. (19) and (21).

The algebraic equations (19) and (21) are $(n+1)$ relations between $(2n+2)$ functions, and obviously, the functions ξ_s, τ, G, μ_s are not unique. For example, corresponding to the same conserved quantity, we can obtain different integrating factors by the appropriate selection of the functions ξ_s, τ, G and μ_s.

4 An example

Suppose that a material point with mass m moves in a vertical plane not below a smooth curve $y = x$, and is subject to non-potential forces $Q_1'' = m\dot{x}$, $Q_2'' = -m\dot{y}$. Let us try to study its conservation law.

Firstly, we study the direct problem to obtain the conserved quantity of the system. We may take $q_1 = x$, $q_2 = y$ as the generalized coordinates. The Lagrangian function of the system and the unilateral constraint are

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - mgq_2,$$
$$f_1 = q_2 - q_1 \geqslant 0, \tag{22}$$

and the non-potential forces are

$$Q_1'' = m\dot{q}_1, \quad Q_2'' = -m\dot{q}_2. \tag{23}$$

The differential equations of motion of the system can be expressed as follows:

$$\ddot{q}_1 = (-g + \dot{q}_1 - \dot{q}_2)/2,$$
$$\ddot{q}_2 = (-g + \dot{q}_1 - \dot{q}_2)/2, \quad f_1 = 0, \tag{24}$$

$$\ddot{q}_1 = \dot{q}_1, \quad \ddot{q}_2 = -g - \dot{q}_2, \quad f_1 > 0. \tag{25}$$

In this problem, the hypothesis of smoothness gives a connecting condition of the system, which depicts the properties within the tangent plane of the constraint hypersurface. Since the property along the normal plane of the constraint hypersurface is still unknown, the motion of the system can not be determined completely.

The generalized Killing equations (14) ~ (16) give

$$\begin{aligned}
&-mg\xi_2 + \frac{1}{2}(-mg + m\dot{q}_1 - m\dot{q}_2)(\xi_1 - \dot{q}_1\tau)\\
&\quad + \frac{1}{2}(mg + m\dot{q}_1 - m\dot{q}_2)(\xi_2 - \dot{q}_2\tau)\\
&\quad + m\dot{q}_1\left(\frac{\partial\xi_1}{\partial t} + \frac{\partial\xi_1}{\partial q_1}\dot{q}_1 + \frac{\partial\xi_1}{\partial q_2}\dot{q}_2\right)\\
&\quad + m\dot{q}_2\left(\frac{\partial\xi_2}{\partial t} + \frac{\partial\xi_2}{\partial q_1}\dot{q}_1 + \frac{\partial\xi_2}{\partial q_2}\dot{q}_2\right)\\
&\quad - (m(\dot{q}_1^2 + \dot{q}_2^2)/2 + mgq_2)\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q_1}\dot{q}_1 + \frac{\partial\tau}{\partial q_2}\dot{q}_2\right)\\
&\quad - \frac{\partial G}{\partial t} - \frac{\partial G}{\partial q_1}\dot{q}_1 - \frac{\partial G}{\partial q_2}\dot{q}_2\\
&\quad - \frac{1}{2}(-mg + m\dot{q}_1 - m\dot{q}_2)\mu_1\\
&\quad - \frac{1}{2}(-mg + m\dot{q}_1 - m\dot{q}_2)\mu_2 = 0, \quad f_1 = 0;
\end{aligned} \tag{26}$$

$$\begin{aligned}
&-mg\xi_2 + m\dot{q}_1(\xi_1 - \dot{q}_1\tau) - m\dot{q}_2(\xi_2 - \dot{q}_2\tau)\\
&\quad + m\dot{q}_1\left(\frac{\partial\xi_1}{\partial t} + \frac{\partial\xi_1}{\partial q_1}\dot{q}_1 + \frac{\partial\xi_1}{\partial q_2}\dot{q}_2\right)\\
&\quad + m\dot{q}_2\left(\frac{\partial\xi_2}{\partial t} + \frac{\partial\xi_2}{\partial q_1}\dot{q}_1 + \frac{\partial\xi_2}{\partial q_2}\dot{q}_2\right)\\
&\quad - (m(\dot{q}_1^2 + \dot{q}_2^2)/2 + mgq_2)\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q_1}\dot{q}_1 + \frac{\partial\tau}{\partial q_2}\dot{q}_2\right)\\
&\quad - \frac{\partial G}{\partial t} - \frac{\partial G}{\partial q_1}\dot{q}_1 - \frac{\partial G}{\partial q_2}\dot{q}_2\\
&\quad - m\dot{q}_1\mu_1 + (mg + m\dot{q}_2)\mu_2 = 0, \quad f_1 > 0;
\end{aligned} \tag{27}$$

$$\begin{aligned}
&m\dot{q}_1\frac{\partial\xi_1}{\partial \dot{q}_1} + m\dot{q}_2\frac{\partial\xi_2}{\partial \dot{q}_1} - (m(\dot{q}_1^2 + \dot{q}_2^2)/2 + mgq_2)\frac{\partial\tau}{\partial \dot{q}_1}\\
&\quad - \frac{\partial G}{\partial \dot{q}_1} + m\mu_1 = 0;
\end{aligned} \tag{28}$$

$$\begin{aligned}
&m\dot{q}_1\frac{\partial\xi_1}{\partial \dot{q}_2} + m\dot{q}_2\frac{\partial\xi_2}{\partial \dot{q}_2} - (m(\dot{q}_1^2 + \dot{q}_2^2)/2 + mgq_2)\frac{\partial\tau}{\partial \dot{q}_2}\\
&\quad - \frac{\partial G}{\partial \dot{q}_2} + m\mu_2 = 0.
\end{aligned} \tag{29}$$

Eqs. (26) ~ (29) have a solution of

$$\tau = 0, \quad \xi_1 = 1, \quad \xi_2 = 1,$$
$$G = mq_1 - mq_2 - mgt, \quad \mu_1 = 0, \quad \mu_2 = 0. \tag{30}$$

In correspondence with the set of functions (30), and according to Theorems 1 and 2, the system has a conserved quantity as follows:

$$I = m\mathring{q}_1 + m\mathring{q}_2 - mq_1 + mq_2 + mgt = \text{const.} \tag{31}$$

Secondly, we study the inverse problem, in which integrating factors ξ_s and functions τ, G, μ_s can be obtained from a given integral. Suppose the system has a first integral (31), then Eqs. (19) and (21) give respectively

$$\xi_1 = 1 + \mathring{q}_1\tau + \mu_1, \quad \xi_2 = 1 + \mathring{q}_2\tau + \mu_2, \tag{32}$$

$$G = m\mathring{q}_1(1 + \mu_1) + m\mathring{q}_2(1 + \mu_2) + L\tau - m\mathring{q}_1 - m\mathring{q}_2 + mq_1 - mq_2 - mgt. \tag{33}$$

There are 6 unknown functions in Eqs. (32) and (33), therefore, the solutions are not unique. We can get the remaining 3 functions by the appropriate selection of 3 functions among the unknown functions. For example, we select

$$\tau = 0, \quad \mu_1 = -1, \quad \mu_2 = 0, \tag{34}$$

then we have

$$\xi_1 = 0, \quad \xi_2 = 1, \quad G = -m\mathring{q}_1 + mq_1 - mq_2 - mgt. \tag{35}$$

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